Resolution of Non-Singularities for Mumford Curves

Emmanuel Lepage

IMJ-PRG, Sorbonne Université

Combinatorial Anabelian Geometry

Outline

- If X hyperbolic curve over finite extension of $\mathbb{Q}_p$, X is a Mumford curve if it has potentially multiplicative reduction.
- Resolution of non-singularities (RNS_X):
  Every semistable model of X is dominated by the stable model of some finite étale cover of X
- Main result of this talk: Mumford curves satisfy RNS.

Berkovich spaces

Let X be an alg. variety $/K$ non-archimedean field.
If $X = \operatorname{Spec} A$,

$$X^{\text{an}} = \{\text{mult. semi} - \text{norms } A \to \mathbb{R}_{\geqslant 0}, \text{extending norm of } K\}$$

topology: coarsest s.th. $\forall f \in A, x := |-(x)| \in X \mapsto |f(x)| \in \mathbb{R}$ cont.
$X \mapsto X^{\text{an}}$ functorial, maps open coverings to open coverings.
$\Longrightarrow$ glues together for general X.
set theor., $X^{\text{an}} = \{(x, |-|); x \in X, |-| : \text{mult. norm on } k(x)\}$
If $x = (x_0, |-|) \in X^{\text{an}}$, $\mathcal{H}(x) := \widehat{k(x_0)}$ for $|-|$.
Example of points:

- $X_{cl} \hookrightarrow X^{an}$ (rigid points)
- $X(\hat{\overline{K}}) \to X^{an}$ (type 1 points ($\supset$ rigid points))
- Assume X smooth. Let $\mathfrak{X}$ normal model of X, Z irred. compon. of special fiber $\mathfrak{X}_s$, then $v_z = mult_Z$ on $K(X)$ induces $\eta_Z \in X^{\text{an}}$.

Example: the affine line

Let $X = \mathrm{Spec}(\mathbb{C}_p[T]) = \mathbb{A}^1_{\mathbb{C}_p}$.

If $a \in \mathbb{C}_p, r \in \mathbb{R}_{\geqslant 0}$, $|\sum_i a_i(T-a)^i|_{b_{a,r}} := \max_i(|a_i|r^i) \quad \rightsquigarrow b_{a,r} \in \mathbb{A}^{1,an}_{\mathbb{C}_p}$

- If r=0, $b_{a,r}$ of type 1.
- If $r \in |p|^{\mathbb{Q}}$, $b_{a,r}$ of type 2.
- If $r \notin |p|^{\mathbb{Q}}$, $b_{a,r}$ of type 3 ($\mathrm{rk}(|K(X)^\times|_{b_{a,r}}) = 2$).
- + points of type 4 corr. to decreasing sequences of balls with empty intersection.

Classification generalizes to analytic curves:

- type 1 if $\mathcal{H}(x) \subset \widehat{\overline{K}}$
- type 2 if $tr.dg.(\widetilde{\mathcal{H}(x)}/\widetilde{K}) = 1$
- type 3 if $rk(|\mathcal{H}(x)^\times|/|K^\times|) = 1$
- type 4 otherwise: $\mathcal{H}(x)$ immediate extension of K (if K alg. cl.)

Berkovich curves

X/K: smooth curve over non-archimedean field
$\overline{X}$: smooth compactification of X
$\mathfrak{X}/O_K$: semi-stable model of X/K
$\mathbb{G}_{\mathfrak{X}}$: dual graph of the semi-stable curve $\mathfrak{X}_s$
There is a natural topological embedding ι and a strong deformation retraction π:

$$\mathbb{G}_{\mathfrak{X}} \underset{\pi}{\overset{\iota}{\rightleftarrows}} X^{\mathrm{an}}$$

$X^{\mathrm{an}} \backslash \iota(\mathbb{G}_{\mathfrak{X}})$: disjoint union of potential open disks (becomes a disk after finite extension of the base field).
By taking the inverse limit over all potential semi-stable models, they induce a homeomorphism

$$\overline{X}^{\mathrm{an}} \xrightarrow{\sim} \varprojlim_{\mathfrak{X}/K'} \mathbb{G}_{\mathfrak{X}}$$

Mumford Curves & Schottky Uniformization

Definition

Let K be a complete non archimedean field.
A smooth curve X over K is a Mumford curve if, after finite base change K'/K, every normalized irreducible component of its stable model $\mathfrak{X}_{O_{K'}}$ is isomorphic to $\mathbb{P}^1$.

Let X/K be a Mumford curve and Ω be the universal cover of X^{an} (as a topological space).

Theorem

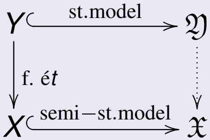
After some finite base change K'/K there is a compact subspace $\mathcal{L} \subset \mathbb{P}^1(K') \subset \mathbb{P}^{1,an}_{K'}$ and an isomorphism of K'-analytic spaces

$$\Omega_{K'} \xrightarrow{\sim} \mathbb{P}^{1,an}_{K'} - \mathcal{L}$$

Resolution of Non-Singularities

Definition

- Let X be a hyperbolic curve over an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. X satisfies resolution of non-singularities (RNS_X) if for every semi-stable model $\mathfrak{X}$ of X, there exists a finite étale cover $f : Y \to X$ such that f extends to a morphism $\mathfrak{Y} \to \mathfrak{X}$ where $\mathfrak{Y}$ is the stable model of Y.



- Let X be a hyperbolic curve over a finite extension $K$of $\mathbb{Q}_p$. X satisfies resolution of non-singularities (RNS_X) if its pullback to an algebraic closure of K does.

Anabelian Motivation

Anab. reconstr. of the graph of st. red $\stackrel{RNS}{\Longrightarrow}$ Anab. reconstr of the graph of arbitrary s.-st. model.

- Abs. Anab. Conj.
- Galois section factor through the decomp. gp of a *unique* Berkovich point
- Reconstr. of Berkovich sp. from geom. tempered fd'l gp
- Retraction of $G_{\mathbb{Q}_p} \to GT_p$ (Tsujimura)

Valuative version of RNS

Let X be a hyperbolic $\mathbb{C}_p$-curve

Definition

- $Sk(X) = \{\eta_Z, Z \text{ irr. comp of the st. reduction of } X\} \subset X^{an}$
- $QSk(X) = \bigcup_{f:Y\to X} f^{an}(Sk(Y)) \subset X^{an}$.

Proposition

TFAE:

- *X satisfies RNS.*
- *$QSk(X)$ is the set of type 2 points of X^{an}.*
- *$QSk(X)$ is dense in X^{an}.*

Stability properties

Theorem

Let X, Y be two hyperbolic curves over $\overline{\mathbb{Q}}_p$ and assume that Y satisfies RNS.

1. *If there is a dominant map $f : X \to Y$, then X satisfies RNS.*
2. *If there is a finite étale cover $f : Y \to X$, then X satisfies RNS.*

Main Result

Theorem

If X is a Mumford curve over a finite extension of $\mathbb{Q}_p$, then it satisfies resolution of non singularities.

Sketch of the proof:

- Step 1: A map $p_X : H^1(X_{\mathbb{C}_p}, \mathbb{Z}_p(1)) \to H^0(X_{\mathbb{C}_p}, \Omega^1)$
- Step 2: Local criterion near type 1 points to get q-sk. points from $\mathbb{Z}_p - torsors$
- Step 3: Rationality of p_X for Mumford curves

A map from p-adic Hodge theory

$X_{ét}$: an. étale topos; $X_{pro-ét}$: an. pro-étale topos
$\nu : X_{pro-ét} \to X_{ét}$
Faltings' exact sequence on $X_{pro-ét}$:

$$0 \to \hat{\mathcal{O}}(1) \to gr^1(\mathcal{O}\mathbb{B}_{dR}^+) \to \hat{\mathcal{O}} \otimes_{\mathcal{O}} \Omega^1 \to 0$$

Boundary map induces isom. on $X_{ét}$:

$$R^1\nu_*\hat{\mathcal{O}}(1) \simeq \Omega^1$$

Induces:

$$p : H^1(X_{\mathbb{C}_p}, \mathbb{Z}_p(1)) \to H^0(X_{\mathbb{C}_p}, \Omega^1)$$

Local description of p_X

Let D small disk in $X_{\mathbb{C}_p}$.
Then $Kum_{D,n} : O^\times(D) \to H^1(D, \mu_n)$ is surjective.
Let $c = (c_n) \in H^1(X_{\mathbb{C}_p}, \mathbb{Z}_p(1)) = \varprojlim H^1(X_{\mathbb{C}_p}, \mu_{p^n})$.
Let $f_n \in O^\times(D)$ s.t. $Kum_{D,n}(f_n) = c_{n|D}$.
Then $(\frac{df_n}{f_n})_n$ converges in $\Omega^1(D)$ unif. and

$$p_x(c)_{|D} = \lim_n \frac{df_n}{f_n}.$$

Local criterion

Let $x \in D(\mathbb{C}_p)$.
Let $c \in H^1(X_{\mathbb{C}_p}, \mathbb{Z}_p(1)) \backslash H^1_{top}(X^{an}_{\mathbb{C}_p}, \mathbb{Z}_p(1))$ (so that $p_X(c) \neq 0$).
Let $\phi_{n,c} : Y_{n,c} \overset{\mu_{p^n}}{\to} X$ be the μ_{p^n}-torsor corr. to c.

Lemma

Assume $e := mult_x p_X(c)$ is not of the form $p^k - 1$ for any k. Then, for n big enough, $Y_{n,c}$ has a skeletal point in $\phi^{-1}_{n,c}(D)$.

Sketch of proof:
Let t be a local parameter of D s.t. $t(x) = 0$.
Up to replacing D by a smaller disk, $\exists f \in O^\times(D)$ s.t. $c_{n,D} = Kum_{p^n}(f)$.
Then $p_X(c)_D = \frac{df}{f}$.
Up to changing t, one can assume $f = 1 + t^{e+1}$ and let $y \in O(\phi^{-1}_{n,c}(D))$ s.t. $y^{p^n} = f$.

For $n >> 0$, let $r_n = |p|^{\frac{n+\frac{1}{p-1}}{e+1}}$ (radius of conv. of a p^nth root of f);
Let $b_n = b_{0,r_n} \in D$ (point of type 2);
Let $a \in \mathbb{C}_p$ s.t. $a^p = -p$;
$\exists b'_n \in Y_{n,c}$ s.t. $\phi_{n,c}(b'_n) = b_n$, $|\frac{y-1}{a}|_{b'_n} = 1$ and the reduction z of $\frac{y-1}{a}$ in $\widetilde{\mathcal{H}(b'_n)}$ satisfies an Artin-Schreier equation:

$$z^p - z = u^{e+1}$$

where u is a rational parameter of $\widetilde{\mathcal{H}(b_n)}$ (given by the reduction of ct for some $c \in \mathbb{C}_p$).
$\rightsquigarrow \widetilde{\mathcal{H}(b_n)}[z]$ is rational iff $e+1$ is a power of p.
$\rightsquigarrow \widetilde{\mathcal{H}(b'_n)}$ is not rational and b'_n is skeletal.

Corollary

Assume $p \neq 2$.
If $p_X(c)(x) = 0$, then $x \in \overline{QSk(X)}$

$p : \Omega_{\mathbb{C}_p} \to X_{\mathbb{C}_p}$ universal top. cover
Kummer exact sequence in Berkovich étale topology:

$$\begin{array}{ccccccc} & & & & H^1(X_{\mathbb{C}_p}, \mu_n) & \longrightarrow & H^1(X_{\mathbb{C}_p}, O^\times) \\ & & & \swarrow^{\alpha_n} & \downarrow & & \downarrow 0 \\ 1 & \longrightarrow & O(\Omega)^\times / (O(\Omega_{\mathbb{C}_p})^\times)^n & \longrightarrow & H^1(\Omega_{\mathbb{C}_p}, \mu_n) & \longrightarrow & H^1(\Omega_{\mathbb{C}_p}, O^\times) \end{array}$$

Comm. diagram:

$$\begin{array}{ccccc} \mathbb{C}_p(X_{\mathbb{C}_p})^\times & \longrightarrow & \mathrm{Div}(X_{\mathbb{C}_p}) & \twoheadrightarrow & H^1(X_{\mathbb{C}_p}, O^\times) \\ \downarrow & & \downarrow & & \downarrow 0 \\ \mathcal{M}(\Omega_{\mathbb{C}_p})^\times & \twoheadrightarrow & \mathrm{Div}(\Omega) & \longrightarrow & H^1(\Omega_{\mathbb{C}_p}, O^\times) \end{array}$$

where
$\mathrm{Div}(\Omega_{\mathbb{C}_p}) = \{f : \Omega_{\mathbb{C}_p}(\mathbb{C}_p) \to \mathbb{Z} \,|\, Supp(\{x \in \Omega_{\mathbb{C}_p}(\mathbb{C}_p) f(x) \neq 0\}) \text{ discrete}\}$.
$\mathcal{M}(\Omega_{\mathbb{C}_p})$: meromorphic functions on $\Omega_{\mathbb{C}_p}$.

If $c = (c_n) \in H^1(X_{\mathbb{C}_p}, \mathbb{Z}_p(1)) = \varprojlim H^1(X_{\mathbb{C}_p}, \mu_{p^n})$,
let $f_n \in O^\times(\Omega_{\mathbb{C}_p})$ be a lifting of $\alpha_{p^n}(c_n)$.

$$\frac{df_n}{f_n} \overset{n\to\infty}{\to} p_X(c)_{|\Omega_{\mathbb{C}_p}} \text{ unif. on every compact of } \Omega_{\mathbb{C}_p}$$

$\bigcup_{\text{finite } I\subset\mathcal{L}} O^\times(\mathbb{P}^1_{\mathbb{C}_p}\backslash I)$ is dense in $O^\times(\Omega_{\mathbb{C}_p}) \rightsquigarrow$ if $\mathcal{L} \subset \mathbb{P}^1(K')$ and $y \in \Omega(K')$, then $p_X(c)_{|\Omega_{\mathbb{C}_p}}(y) \in K'dt$.

Let $x \in X(K')$, where K' is a finite extension of K.

$$\begin{array}{ccccc}
H^1(X,\mathbb{Z}_p(1))/H^1_{top}(X,\mathbb{Z}_p(1)) & \hookrightarrow & H^0(X,\Omega^1_X) & \xrightarrow{ev_x} & K' \subset \mathbb{C}_p \\
\downarrow & & \downarrow & \nearrow_{ev_{x'}} & \\
H^1(X',\mathbb{Z}_p(1))/H^1_{top}(X',\mathbb{Z}_p(1)) & \hookrightarrow & H^0(X',\Omega^1_{X'}) & &
\end{array}$$

where X' is any topological finite cover and $x' \in X'(K')$ is a preimage of x.

As $g(X') \to \infty$, $\dim_{\mathbb{Q}_p} H^1(X',\mathbb{Q}_p(1))/(\mathrm{Ker}\, p_{X'}) \to \infty$, but $\dim_{\mathbb{Q}_p} K'$ stays finite

$\implies \exists (X',x')$ and $c \in H^1(X,\mathbb{Z}_p(1)) \backslash (\mathrm{Ker}\, p_{X'})$ s.t. $p_X(c)(x') \neq 0$.

$\rightsquigarrow$ X satisfies RNS if $p \neq 2$.